More generally, how can we determine the basis of a span?

Key Point 1: The pivot columns of a matrix A are linearly independent.

Why? If we put the pivot columns in a matrix and row reduce, we get a pivot in each column, so they are linearly independent

Ex: $A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$

Pivot columns are L.I.

Pivot columns

Key Point 2: Every non pivot column of A is a linear combination of the pivot columns

Why? By defn of RREF, the nonpivot columns of R must be L.Cs of the pivot columns. The same relationship holds for A because row operations don't change linear combination relations.

Ex: In E (above),

Column 3: $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, Column 4: $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

In A, use same weights

Column 3: $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, Column 4: $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Consider Span$\{v_1, \dots, v_p\}$. Let $A = [v_1 \cdots v_p]$.

By Key Point 2, every nonpivot column is a linear combination of the pivot columns.

Hence Span$\{v_1, \dots, v_p\}$ = Span{pivot columns}

Moreover, by Key Point 1 the pivot columns are linearly independent. Hence the pivot columns of A form a basis. That is:

The pivot columns of $A = [v_1 \cdots v_p]$ form a basis for Span$\{v_1, \dots, v_p\}$

Ex: Find a basis for Span$\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}\right\}$.

$\begin{bmatrix} 1 & -2 & 7 \\ 2 & -4 & 8 \\ 3 & -6 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Since the pivot columns are $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$,

$\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}\right\}$ forms a basis.

This tells us that this subspace is a plane in $\mathbb{R}^3$.

From this fact, we now know bases of $C(A)$ and $N(A)$!

Basis for $C(A)$: The pivot columns of A form a basis for $C(A)$

This is because $C(A) = \text{Span}\{\text{columns of } A\}$.

Basis for $N(A)$: The special solutions form a basis for $N(A)$.

This is because the special solutions span $N(A)$ (by construction) and are linearly independent (since there is a pivot in each column of the associated matrix)

Dimension

Key Fact: Any two bases of a subspace must contain the same number of vectors.

Proof:

Let $\{b_1,\dots,b_n\}$ and $\{d_1,\dots,d_m\}$ be bases for V.

We want to show that $n=m$.

Assume $m<n$ (the other case is similar)

Since $\{b_1,\dots,b_n\}$ is a basis, $d_1,\dots,d_m$ can be expressed as linear combination of $\{b_1,\dots,b_n\}$.

That is $d_i = a_{i1}b_1 + a_{i2}b_2 + \cdots + a_{in}b_n \quad \forall\ 1\le i\le m$

Hence $\underbrace{[b_1 \cdots b_n]}_{B} \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{A} = \underbrace{[d_1 \cdots d_n]}_{D}$

Since $m<n$, $Ax=0$ has a nonzero solution since there must be free variables

Hence $BAx=0$ has a nonzero solution

$\Rightarrow$ $Dx=0$ has a nonzero solution

$\Rightarrow$ $\{d_1,\dots,d_n\}$ is not linearly independent

$\Rightarrow$ It is not a basis.

But, it is a basis. So our initial assumption that $m<n$ is incorrect. Hence $m=n$ ∎

This leads to the main concept of this section:

Def: The dimension of a subspace V is the number of vectors in a basis of V.

Ex: $\mathbb{R}^n$ is n-dimensional since it has basis

$\Sigma = \{e_1, e_2, \dots, e_n\}$.

Geometry:

0-dim'l subspace: $\{0\}$

1-dim'l subspace: line through the origin

2-dim'l subspace: plane through the origin

3-dim'l subspace: 3D space through the origin

etc.

Def: The rank of A is the number of pivots of A.

Since the pivot columns of A form a basis of $C(A)$, we have that $\dim C(A) = \text{rank}(A)$.

So if A is $m \times n$, then

$\dim C(A) = \text{rank}(A)$

$\dim N(A) = \#\text{special solns} = \#\text{free variables} = n - \text{rank}(A)$

Rank-Nullity thm: If A is $m \times n$, then $\dim C(A) + \dim N(A) = n$

$\text{rank}A + \text{nullity}(A) = n$